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Introduction

- In August of 1904 Ludwig Prandtl,a 29-year old professor presented a remarkable paper on BOUNDARY LAYER at the 3rd International Mathematical Congress in Heidelberg.

- The condition of zero fluid velocity at the solid surface is referred to as ‘no slip’ and the layer of fluid between the surface and the free stream fluid is termed BOUNDARY LAYER.

Boundary Layer History

- ❖1904 Prandtl

Fluid Motion with Very Small Friction

2-D boundary layer equations

- ❖1908 Blasius

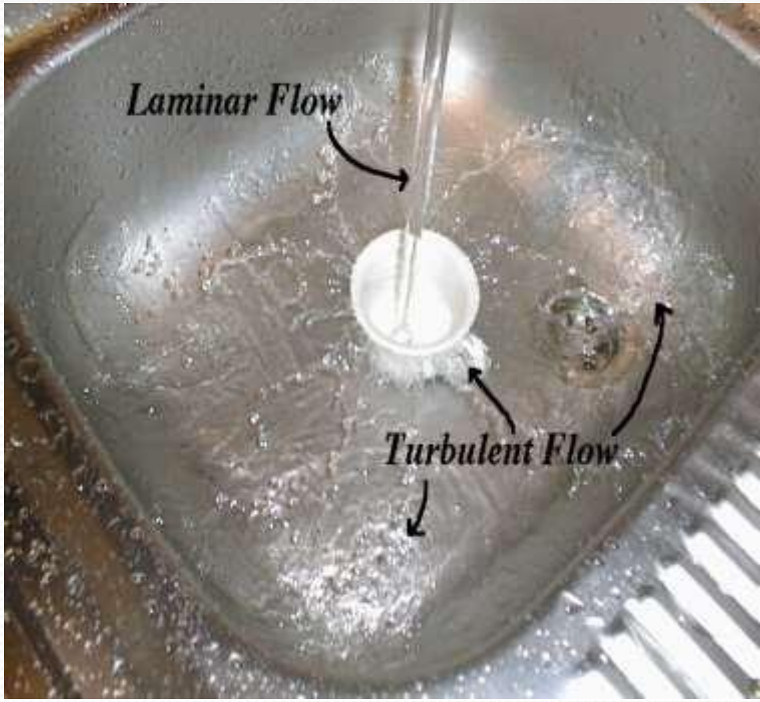
The Boundary Layers in Fluids with Little Friction

Solution for laminar, 0-pressure gradient flow

- ❖1921 von Karman

Integral form of boundary layer equations

Laminar Flow



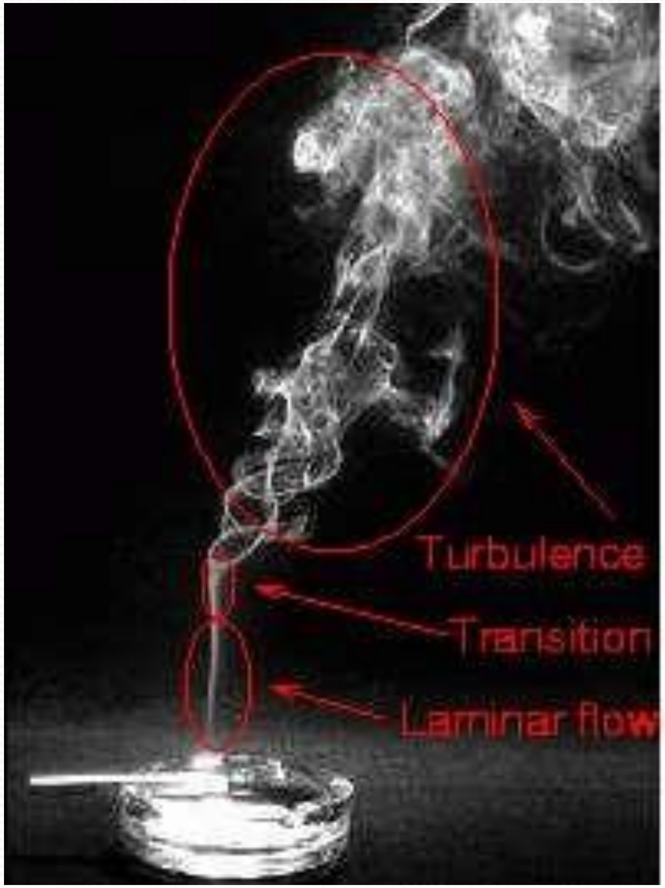
- Each liquid particle has a definite path.
- The paths of individual particles do not cross each other.
- All the molecules in the fluid move in the same direction and at the same speed.
- It also called as stream line flow.

Turbulent Flow

- Each liquid particle do not have a definite path.

- The path of individual particle also cross each other.

- The molecules in the fluid move in different directions and at different speeds.



Critical Velocity

- A velocity at which the flow changes from the laminar flow to turbulent flow.

- The critical velocity may be further classified into the following two types :1.Lower Critical Velocity

  2.Upper Critical Velocity

Reynold's Number

Re =Inertia forces/viscus forces

$=(\rho v^2)/(\mu v/d)$

$=\rho vd/\mu$

$=vd/V$ (as $\rho/\mu=V$)

$$Re=\frac{\text{Mean velocity of liquid} \times \text{Diameter of pipe}}{\text{Kinematic velocity of liquid}}$$

- ❖ Re < 2000 ; Laminar flow
- ❖ 2000 < Re < 2800 ;Transition flow
- ❖ 2800 < Re ; Turbulent flow

Boundary Layer Theory

- A thin layer of fluid acts in such a way ,as if it's inner surface is fixed to the boundary of the body.

- Velocity of flow at boundary layer is zero.

- The velocity of flow will go on increasing rapidly till at the extreme layer.

- The portion which is outside the boundary layer has a high value of Reynold's Number, because of the high velocity of flow

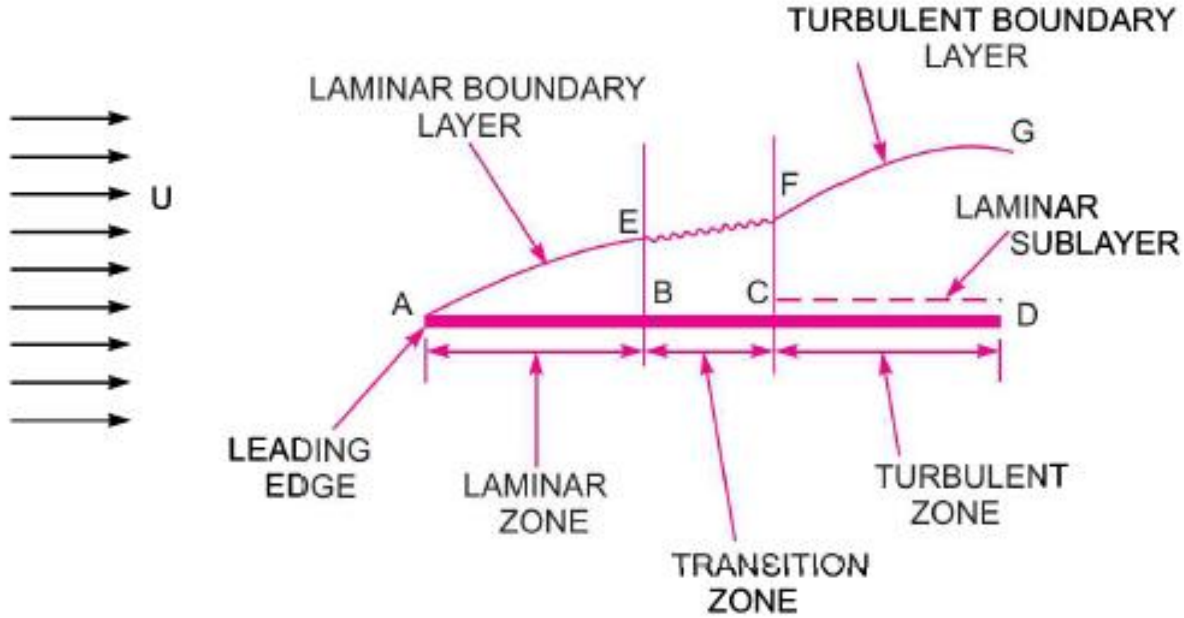
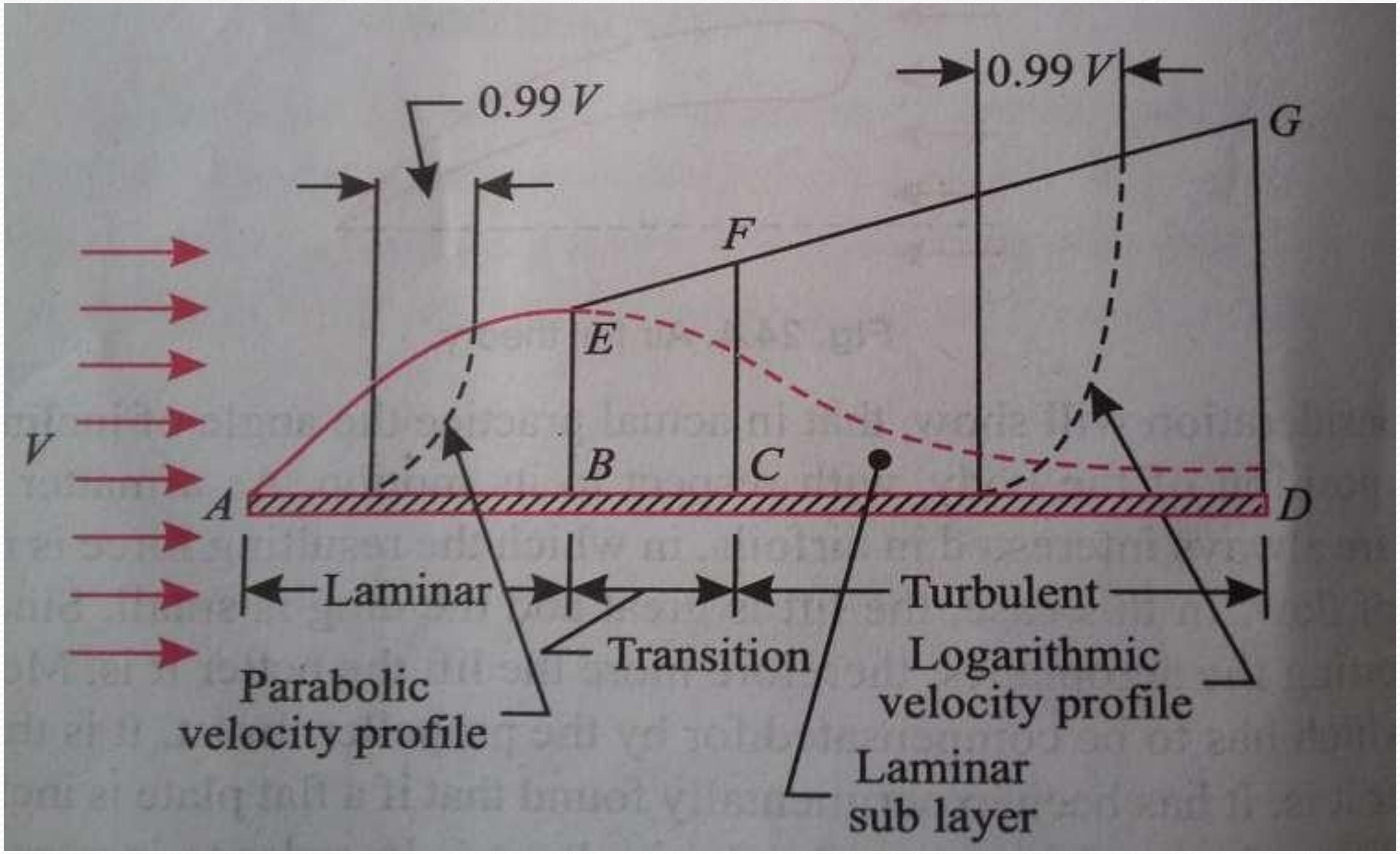


Fig. 13.2 *Flow over a plate.*

Hence for laminar boundary layer, we have $5 \times 10^5 = \dfrac{U \times x}{\nu}$

Thickness Of Boundary Layer



- The distance from surface of the body ,to a place where the velocity of flow is 0.99 times of the maximum velocity of flow ,is known as thickness of boundary layer.

- It is usually denoted by δ(delta).

- R_{NX}= Vx/v

  where, V=Velocity of fluid

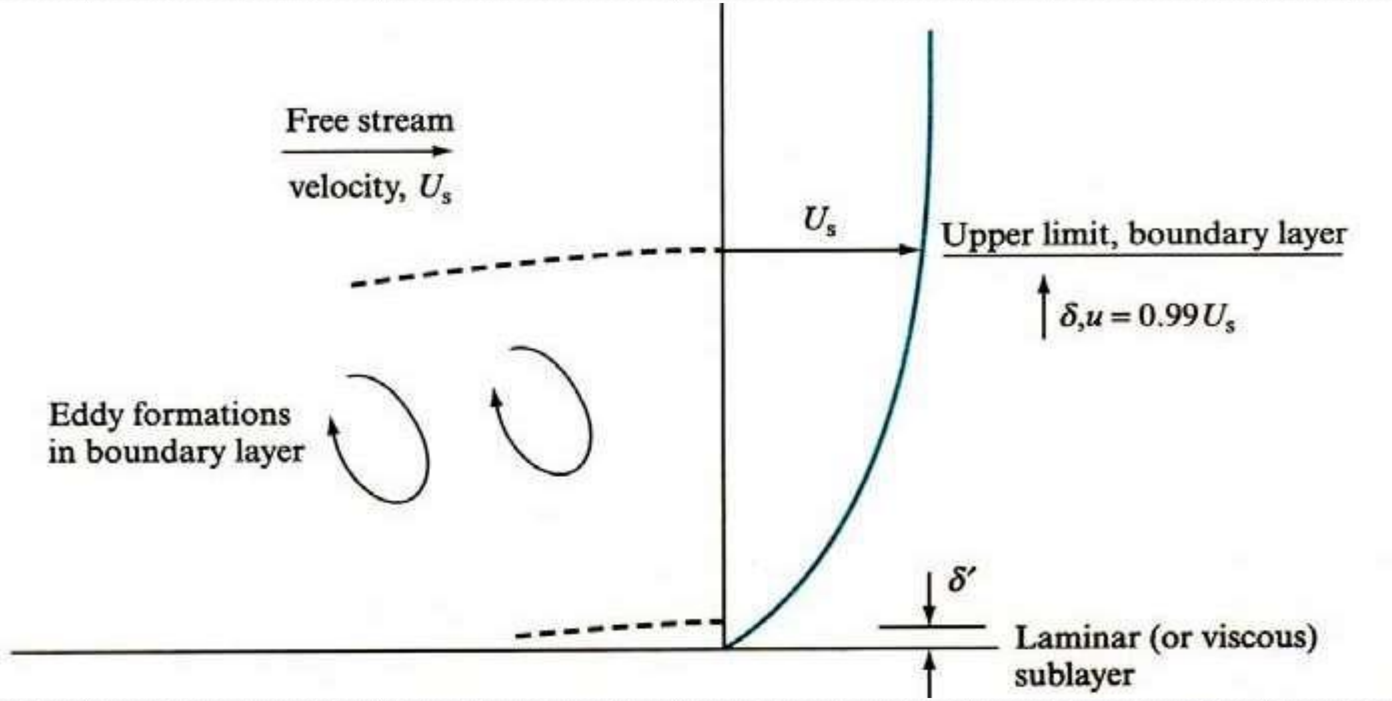
  v=Kinematic velocity of fluid

  x= Distance b/w the leading edge of the plate and the section

13.2.3 Laminar Sub-layer. This is the region in the turbulent boundary layer zone, adjacent to the solid surface of the plate as shown in Fig. 13.2. In this zone, the velocity variation is influenced only by viscous effects. Though the velocity distribution would be a parabolic curve in the laminar sub-layer zone, but in view of the very small thickness we can reasonably assume that velocity variation is linear and so the velocity gradient can be considered constant. Therefore, the shear stress in the laminar sub-layer would be constant and equal to the boundary shear stress τ_0. Thus the shear stress in the sub-layer is

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{u}{y} \qquad \left\{\because \text{ For linear variation, } \frac{\partial u}{\partial y} = \frac{u}{y}\right\}$$

Boundary Layer Thickness, δ



Boundary layer thickness is defined as that distance from the surface where the local velocity equals 99% of the free stream velocity.

$$\delta = y_{(u=0.99U_s)}$$

Thickness Of Boundary Layer In A Laminar Flow

- It has been experimentally found, that the thickness of the boundary layer is zero at the leading edge A, and increases to the trailing edge , the flow is laminar.

- According to Pohlhausenin

$$\delta_{lam} = 5.835x/\sqrt{Rnx}$$

- According to Prandtl-Blassius

$$\delta_{lam} = 5x/\sqrt{Rnx}$$

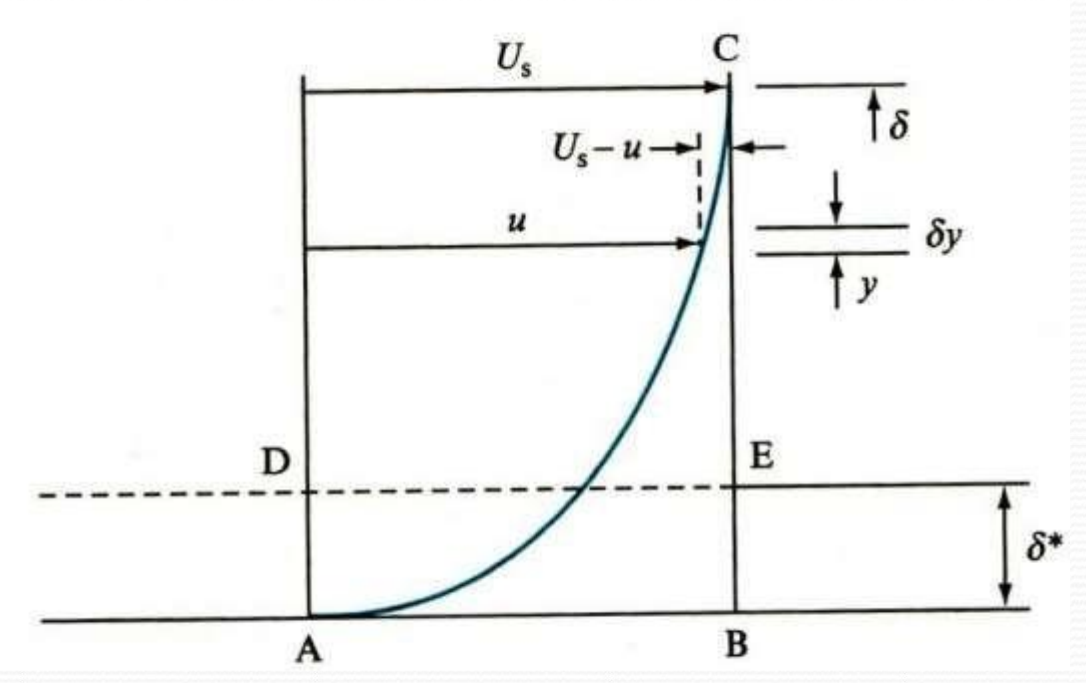
Thickness Of Boundary Layer In A Turbulent Flow

- As the boundary layer continuous further downstream, it expands and the transition flow changes into turbulent flow and the transition boundary layer changes into turbulent boundary layer, which continuous over the remaining length of the plate.

- According to Prandtl-Blassious,

$$\delta_{tur} = 0.377x/(R_{N}x)^{1/5}$$

Displacement Thickness, δ*



The displacement thickness for the boundary layer is defined as the distance the surface would have to move in the y-direction to reduce the flow passing by a volume equivalent to the real effect of the boundary layer.

$$\delta^* = \int_0^\delta (1 - u/Us)\,dy$$

Momentum Thickness, θ

Momentum thickness is the distance that, when multiplied by the square of the free stream velocity, equals the integral of the momentum defect. Alternatively, the total loss of momentum flux is equivalent to the removal of momentum through a distance θ. It is a theoretical length scale to quantify the effects of fluid viscosity near a physical boundary.

$$\theta = \int_0^{\delta} u/Us(1 - u/Us)\,\mathrm{dy}$$

Problem 13.1 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = \frac{y}{\delta}$*, where u is the velocity at a distance y from the plate and u = U at y = δ, where δ = boundary layer thickness. Also calculate the value of δ*/θ.*

Solution. Given :

Velocity distribution $\frac{u}{U} = \frac{y}{\delta}$

(*i*) Displacement thickness δ* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy \qquad \left\{\because \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \left[y - \frac{y^2}{2\delta} \right]_0^\delta \qquad \{\delta \text{ is constant across a section}\}$$

$$= \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2} = \frac{\boldsymbol{\delta}}{\mathbf{2}}. \quad \textbf{Ans.}$$

(*ii*) Momentum thickness, θ is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = \frac{y}{\delta}$,

$$\theta = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{3\delta - 2\delta}{6} = \mathbf{\frac{\delta}{6}}. \textbf{ Ans.}$$

(*iii*) Energy thickness δ^{**} is given by equation (13.6), as

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy = \int_0^\delta \frac{y}{\delta}\left[1 - \frac{y^2}{\delta^2}\right] dy \qquad \left\{\because \quad \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \int_0^\delta \left[\frac{y}{\delta} - \frac{y^3}{\delta^3}\right] dy = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2} - \frac{\delta}{4} = \frac{2\delta - \delta}{4} = \mathbf{\frac{\delta}{4}}. \textbf{ Ans.}$$

(*iv*)
$$\frac{\delta^*}{\theta} = \frac{\left(\frac{\delta}{2}\right)}{\left(\frac{\delta}{6}\right)} = \frac{\delta}{2} \times \frac{6}{\delta} = \mathbf{3}. \textbf{ Ans.}$$

Problem 13.2 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$.

Solution. Given :

Velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

(*i*) Displacement thickness δ^* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$, we have

$$\delta^* = \int_0^\delta \left\{1 - \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right\} dy$$

$$= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy = \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta$$

$$= \delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2} = \delta - \delta + \frac{\delta}{3} = \frac{\delta}{3}. \quad \textbf{Ans.}$$

(*ii*) Momentum thickness θ, is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U}\left\{1 - \frac{u}{U}\right\} dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left[1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]\left[1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy = \left[\frac{2y^2}{2\delta} - \frac{5y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{\delta^2}{\delta} - \frac{5\delta^3}{3\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4}\right] = \delta - \frac{5\delta}{3} + \delta - \frac{\delta}{5}$$

$$= \frac{15\delta - 25\delta + 15\delta - 3\delta}{15} = \frac{30\delta - 28\delta}{15} = \mathbf{\frac{2\delta}{15}}\,. \quad \textbf{Ans.}$$

(*iii*) Energy thickness δ** is given by equation (13.6),

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1-\frac{u^2}{U^2}\right]dy = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]^2\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{4y^2}{\delta^2}+\frac{y^4}{\delta^4}-\frac{4y^3}{\delta^3}\right]\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{4y^2}{\delta^2}-\frac{y^4}{\delta^4}+\frac{4y^3}{\delta^3}\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{8y^3}{\delta^3}-\frac{2y^5}{\delta^5}+\frac{8y^4}{\delta^4}-\frac{y^2}{\delta^2}+\frac{4y^4}{\delta^4}+\frac{y^6}{\delta^6}-\frac{4y^5}{\delta^5}\right)dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}-\frac{8y^3}{\delta^3}+\frac{12y^4}{\delta^4}-\frac{6y^5}{\delta^5}+\frac{y^6}{\delta^6}\right]dy$$

$$= \left[\frac{2y^2}{2\delta}-\frac{y^3}{3\delta^2}-\frac{8y^4}{4\delta^3}+\frac{12y^5}{5\delta^4}-\frac{6y^6}{6\delta^5}+\frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \frac{\delta^2}{\delta}-\frac{\delta^3}{3\delta^2}-\frac{2\delta^4}{\delta^3}+\frac{12\delta^5}{5\delta^4}-\frac{\delta^6}{\delta^5}+\frac{\delta^7}{7\delta^6} = \delta-\frac{\delta}{3}-2\delta+\frac{12}{5}\delta-\delta+\frac{\delta}{7}$$

$$= -2\delta-\frac{\delta}{3}+\frac{12}{5}\delta+\frac{\delta}{7} = \frac{-210\delta-35\delta+252\delta+15\delta}{105}$$

$$= \frac{-245\delta+267\delta}{105} = \mathbf{\frac{22\delta}{105}}.$$ **Ans.**

▶ 13.3 DRAG FORCE ON A FLAT PLATE DUE TO BOUNDARY LAYER

Consider the flow of a fluid having free-stream velocity equal to U, over a thin plate as shown in Fig. 13.4. The drag force on the plate can be determined if the velocity profile near the plate is known. Consider a small length Δx of the plate at a distance of x from the leading edge as shown in Fig. 13.4 (a). The enlarged view of the small length of the plate is shown in Fig. 13.4 (b).

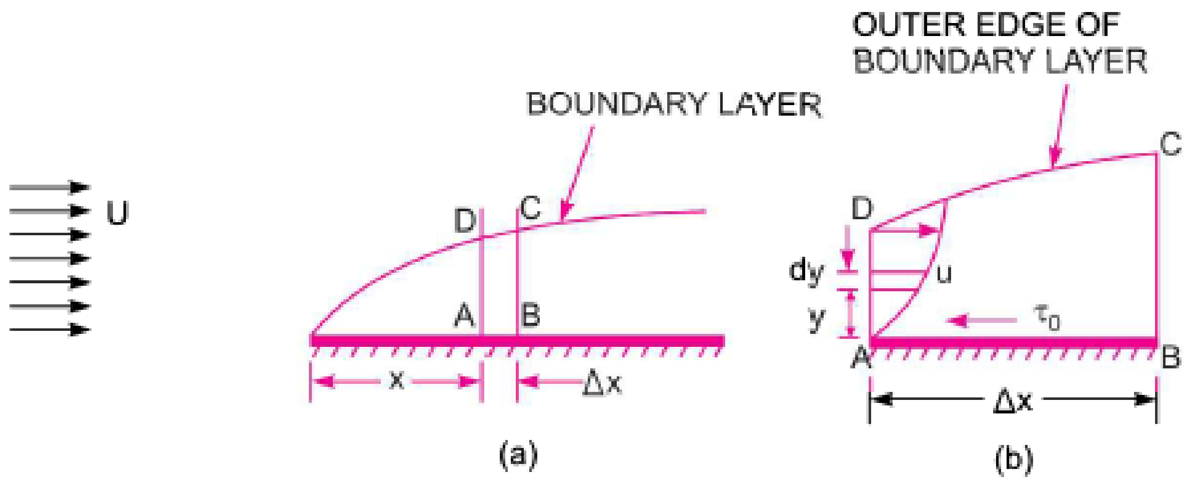


Fig. 13.4 *Drag force on a plate due to boundary layer.*

The shear stress τ_0 is given by $\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0}$, where $\left(\frac{du}{dy}\right)_{y=0}$ is the velocity distribution near the plate at $y = 0$.

Then drag force or shear force on a small distance Δx is given by

$$\Delta F_D = \text{shear stress} \times \text{area}$$

$$= \tau_0 \times \Delta x \times b \quad \text{...(13.7)} \quad \{\text{Taking width of plate} = b\}$$

where ΔF_D = drag force on distance Δx

The drag force ΔF_D must also be equal to the rate of change of momentum over the distance Δx.

Consider the flow over the small distance Δx. Let $ABCD$ is the control volume of the fluid over the distance Δx as shown in Fig. 13.4 (b). The edge DC represents the outer edge of the boundary layer.

Let u = velocity at any point within the boundary layer

b = width of plate

Then mass rate of flow entering through the side AD

$$= \int_0^\delta \rho \times \text{velocity} \times \text{area of strip of thickness } dy$$

$$= \int_0^\delta \rho \times u \times b \times dy \qquad \{\because \text{ Area of strip} = b \times dy\}$$

$$= \int_0^\delta \rho u b dy$$

Mass rate of flow leaving the side BC

$$= \text{mass through } AD + \frac{\partial}{\partial x}(\text{mass through } AD) \times \Delta x$$

$$= \int_0^\delta \rho u b dy \, \frac{\partial}{\partial x}\left[\int_0^\delta (\rho u b dy)\right] \times \Delta x$$

From continuity equation for a steady incompressible fluid flow, we have

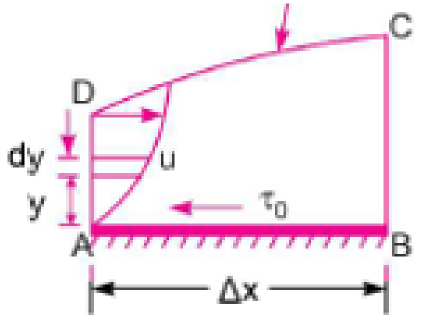
Mass rate of flow entering AD + mass rate of flow entering DC

= mass rate of flow leaving BC

$\because$ Mass rate of flow entering DC = mass rate of flow through BC – mass rate of flow through AD

$$= \int_0^\delta \rho u b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x - \int_0^\delta \rho u b dy$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x$$



(b)

The fluid is entering through side DC with a uniform velocity U.

Now let us calculate momentum flux through control volume.

Momentum flux entering through AD

$$= \int_0^\delta \text{momentum flux through strip of thickness } dy$$

$$= \int_0^\delta \text{mass through strip} \times \text{velocity} = \int_0^\delta (\rho u b dy) \times u = \int_0^\delta \rho u^2 b dy$$

Momentum flux leaving the side $BC = \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x$

Momentum flux entering the side DC = mass rate through DC × velocity

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u b dy\right] \times \Delta x \times U \qquad (\because \text{ Velocity} = U\}$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

As U is constant and so it can be taken inside the differential and integral.

$\therefore$ Rate of change of momentum of the control volume

= Momentum flux through BC – Momentum flux through AD – momentum flux through DC

$$= \int_0^\delta \rho u^2 b dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x - \int_0^\delta \rho u^2 b dy - \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy\right] \times \Delta x - \frac{\partial}{\partial x}\left[\int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 b dy - \int_0^\delta \rho u U b dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\rho u^2 b - \rho u U b\right) dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\rho b \int_0^\delta \left(u^2 - uU\right) dy\right] \times \Delta x$$

{For incompressible fluid ρ is constant}

$$= \rho b \frac{\partial}{\partial x}\left[\int_0^{\delta} \left(u^2 - uU\right)dy\right] \times \Delta x \qquad ...(13.8)$$

Now the rate of change of momentum on the control volume *ABCD* must be equal to the total force on the control volume in the same direction according to the momentum principle. But for a flat plate $\frac{\partial p}{\partial x} = 0$, which means there is no external pressure force on the control volume. Also the force on the side *DC* is negligible as the velocity is constant and velocity gradient is zero approximately. The only external force acting on the control volume is the shear force acting on the side *AB* in the direction from *B* to *A* as shown in Fig. 13.4 (*b*). The value of this force is given by equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

∴ Total external force in the direction of rate of change of momentum

$$= -\tau_0 \times \Delta x \times b \qquad ...(13.9)$$

According to momentum principle, the two values given by equations (13.9) and (13.8) should be the same.

$$\therefore \qquad -\tau_0 \times \Delta x \times b = \rho b \frac{\partial}{\partial x}\left[\int_0^{\delta} \left(u^2 - uU\right)dy\right] \times \Delta x$$

Cancelling $\Delta x \times b$, to both sides, we have

$$-\tau_0 = \rho \frac{\partial}{\partial x}\left[\int_0^{\delta} \left(u^2 - uU\right)dy\right]$$

or

$$\tau_0 = -\rho \frac{\partial}{\partial x}\left[\int_0^{\delta} \left(u^2 - uU\right)dy\right] = \rho \frac{\partial}{\partial x}\left[\int_0^{\delta} \left(uU - u^2\right)dy\right]$$

$$= \rho \frac{\partial}{\partial x}\left[\int_0^{\delta} U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right] = \rho U^2 \frac{\partial}{\partial x}\left[\int_0^{\delta} \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

or

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^{\delta} \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right] \qquad ...(13.10)$$

In equation (13.10), the expression $\int_0^{\delta} \frac{u}{U}\left[1 - \frac{u}{U}\right]dy$ is equal to momentum thickness θ. Hence equation (13.10) is also written as

$$\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x} \qquad ...(13.11)$$

Equation (13.11) is known as **Von Karman momentum integral equation** for boundary layer flows.

For a given velocity profile in laminar zone, transition zone or turbulent zone of a boundary layer, the shear stress τ_0 is obtained from equation (13.10) or (13.11). Then drag force on a small distance Δx of the plate is obtained from equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

Then total drag on the plate of length L on one side is

$$F_D = \int \Delta F_D = \int_0^L \tau_0 \times b \times dx \quad \{\text{change } \Delta x = dx\}. \qquad \text{...(13.12)}$$

13.3.1 Local Co-efficient of Drag [C_D*].

It is defined as the ratio of the shear stress τ_0 to the quantity $\frac{1}{2}\rho U^2$. It is denoted by C_D*

Hence
$$C_D^* = \frac{\tau_0}{\frac{1}{2}\rho U^2}. \qquad \text{...(13.13)}$$

13.3.2 Average Co-efficient of Drag [C_D].

It is defined as the ratio of the total drag force to the quantity $\frac{1}{2}\rho A U^2$. It is also called co-efficient of drag and is denoted by C_D.

Hence
$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2} \qquad \text{...(13.14)}$$

where A = Area of the surface (or plate)

U = Free-stream velocity

ρ = Mass density of fluid.

Problem 13.3 *For the velocity profile for laminar boundary layer flows given as*

$$\frac{u}{U} = 2(y/\delta) - (y/\delta)^2$$

find an expression for boundary layer thickness (δ), shear stress (τ_0) and co-efficient of drag (C_D) in terms of Reynold number.

Solution. Given :

(*i*) The velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ ...(*i*)

Substituting this value of $\frac{u}{U}$ in equation (13.10), we get

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]\left[1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right] dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]\left[1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right] dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy\right]$$

$$= \frac{\partial}{\partial x}\int_0^\delta \left[\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy = \frac{\partial}{\partial x}\left[\frac{2y^2}{2\delta} - \frac{5 \times y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{\delta^2}{\delta} - \frac{5}{3}\frac{\delta^3}{\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4}\right] = \frac{\partial}{\partial x}\left[\delta - \frac{5}{3}\delta + \delta - \frac{\delta}{5}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{15\delta - 25\delta + 15\delta - 3\delta}{15}\right] = \frac{\partial}{\partial x}\left[\frac{30\delta - 28\delta}{15}\right] = \frac{\partial}{\delta x}\left[\frac{2\delta}{15}\right] = \frac{2}{15}\frac{\partial}{\partial x}[\delta]$$

$$\tau_0 = \rho U^2 \times \frac{2}{15}\frac{\partial}{\partial x}[\delta] = \frac{2}{15}\rho U^2 \frac{\partial[\delta]}{\partial x} \qquad ...(13.15)$$

The shear stress at the boundary in laminar flow is also given by Newton's law of viscosity as

$$\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0} \quad \text{...(ii)}$$

But from equation (*i*), $$u = U\left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]$$

$$\therefore \quad \frac{du}{dy} = U\left[\frac{2}{\delta} - \frac{2y}{\delta^2}\right] \qquad \{\because \; U \text{ is constant}\}$$

$$\therefore \quad \left(\frac{du}{dy}\right)_{y=0} = U\left[\frac{2}{\delta} - \frac{2\times(0)}{\delta^2}\right] = \frac{2U}{\delta}$$

Substituting this value in (*ii*), we get

$$\tau_0 = \mu \times \frac{2U}{\delta} = \frac{2\mu U}{\delta} \quad \text{...(iii)}$$

Equating the two values of τ_0 given by equation (13.15) and (*iii*)

$$\frac{2}{15}\rho U^2 \frac{\partial}{\partial x}[\delta] = \frac{2\mu U}{\delta}$$

or $$\frac{\delta\partial}{\partial x}[\delta] = \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U} \quad \text{or} \quad \delta\partial[\delta] = \frac{15\mu}{\rho U}\partial x$$

As the boundary layer thickness (δ) is a function of *x* only.

Hence partial derivative can be changed to total derivative

$$\therefore \qquad \delta d[\delta] = \frac{15\mu}{\rho U}\, dx$$

On integration, we get
$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U}\, x + C \qquad \left\{\frac{\mu}{\rho U} \text{ is constant}\right\}$$

$x = 0$, $\delta = 0$ and hence $C = 0$

$$\therefore \qquad \frac{\delta^2}{2} = \frac{15\mu x}{\rho U}$$

$$\therefore \qquad \delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = \sqrt{\frac{30\mu x}{\rho U}} = 5.48\sqrt{\frac{\mu x}{\rho U}} \qquad \text{...(13.16)}$$

$$= 5.48\sqrt{\frac{\mu x \times x}{\rho U \times x}} = 5.48\sqrt{\frac{x^2}{R_{e_x}}} \qquad \left\{\because \quad R_{e_x} = \frac{\rho U x}{\mu}\right\}$$

$$= 5.48\,\frac{x}{\sqrt{R_{e_x}}} \qquad \text{...(13.17)}$$

In equation (13.16), μ, ρ and U are constant and hence it is clear from this equation that thickness of laminar boundary layer is proportional to the square root of the distance from the leading edge. Equation (13.17) gives the thickness of laminar boundary layer in terms of Reynolds number.

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(*ii*) **Shear stress (τ_0) in terms of Reynolds number**

From equation (*iii*), we have $\tau_0 = \dfrac{2\mu U}{\delta}$

Substituting the value of δ from equation (13.17), in the above equation, we get

$$\tau_0 = \frac{2\mu U}{5.48\dfrac{x}{\sqrt{R_{e_x}}}} = \frac{2\mu U\sqrt{R_{e_x}}}{5.48x} = 0.365\,\frac{\mu U}{x}\sqrt{R_{e_x}}$$

(*iii*) **Co-efficient of Drag (C_D)**

From equation (13.14), we have $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

where F_D is given by equation (13.12) as

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.365\frac{\mu U}{x}\sqrt{R_{e_x}} \times b \times dx$$

$$= 0.365\int_0^L \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times b \times dx \qquad \left\{\because\ R_{e_x} = \frac{\rho U x}{\mu}\right\}$$

$$= 0.365\int_0^L \mu U\sqrt{\frac{\rho U}{\mu}} \times \frac{1}{\sqrt{x}} \times b \times dx$$

$$= 0.365\,\mu U \sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2}\,dx = 0.365\,\mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^L$$

$$= 0.365 \times 2\mu U \sqrt{\frac{\rho U}{\mu}} \times b \times \sqrt{L}$$

$$= 0.73\, b\mu U \sqrt{\frac{\rho U L}{\mu}} \qquad \text{...(13.18)}$$

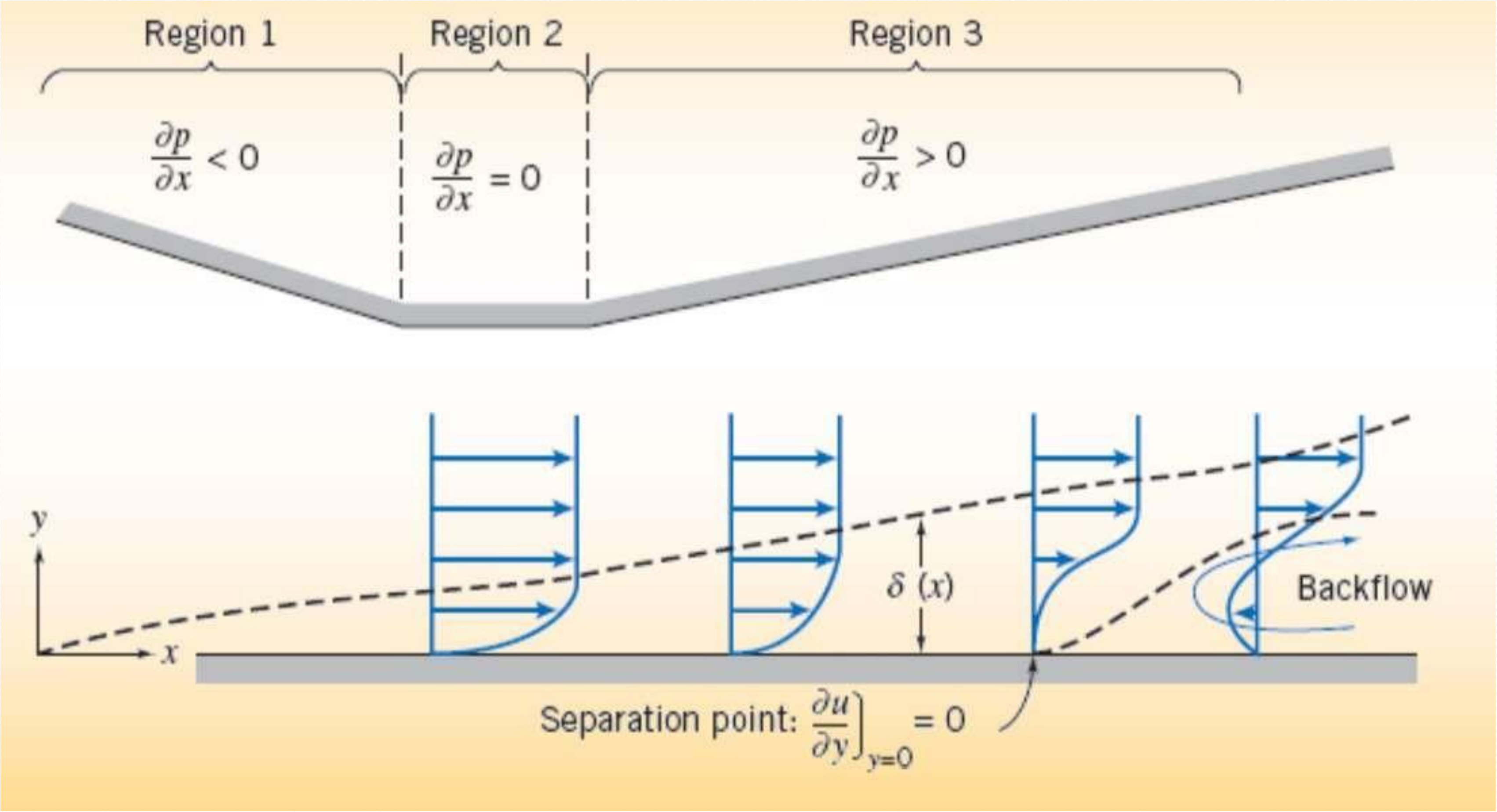
$$\therefore \qquad C_D = \frac{0.73\, b\mu U \sqrt{\frac{\rho U L}{\mu}}}{\frac{1}{2}\rho A U^2}$$

where A = Area of plate = Length of plate × width = $L \times b$

$$\therefore \qquad C_D = \frac{0.73\, b\mu U}{\frac{1}{2}\rho \times L \times b \times U^2}\sqrt{\frac{\rho U L}{\mu}} = \frac{1.46\,\mu}{\rho L U}\sqrt{\frac{\rho U L}{\mu}}$$

$$= \frac{1.46\sqrt{\mu}}{\sqrt{\rho U L}} = 1.46\sqrt{\frac{\mu}{\rho U L}} = \frac{1.46}{\sqrt{R_{e_L}}} \qquad \text{...(13.19)} \qquad \left\{\because \sqrt{\frac{\mu}{\rho U L}} = \frac{1}{\sqrt{R_{e_L}}}\right\}$$

Pressure Gradients In Boundary Layer Flow



Applications Of Boundary Layer Theory

- *Aerodynamics* (Airplanes, Rockets, Projectiles)
- *Hydrodynamics* (Ships, Submarines, Torpedoes)
- *Transportation* (Automobiles, Trucks, Cycles)
- *Wind Engineering* (Buildings, Bridges, Water Towers)
- *Ocean Engineering* (Buoys, breakwaters, Cables).

Velocity Distribution		δ	C_D
1.	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$	$5.48\ x/\sqrt{R_{e_x}}$	$1.46/\sqrt{R_{e_L}}$
2.	$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	$4.64\ x/\sqrt{R_{e_x}}$	$1.292/\sqrt{R_{e_L}}$
3.	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$	$5.84\ x/\sqrt{R_{e_x}}$	$1.36/\sqrt{R_{e_L}}$
4.	$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	$4.79\ x/\sqrt{R_{e_x}}$	$1.31/\sqrt{R_{e_L}}$
5.	Blasius's Solution	$4.91\ x/\sqrt{R_{e_x}}$	$1.328/\sqrt{R_{e_L}}$

Problem 13.7 *For the velocity profile for laminar boundary flow* $\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$. *Obtain an expression for boundary layer thickness, shear stress, drag force on one side of the plate and co-efficient of drag in terms of Reynold number.*

Solution. (*i*) The velocity profile is $\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$.

Substituting this value in equation (13.10), we have

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\left[1-\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\left[\frac{-\cos\frac{\pi y}{2\delta}}{\frac{\pi}{2\delta}}\right] - \left[\frac{\frac{\pi y}{2\delta}\times\frac{1}{2}}{\frac{\pi}{2\delta}} - \frac{\sin 2\left(\frac{\pi}{2}\frac{y}{\delta}\right)}{4\times\frac{\pi}{2\delta}}\right]\right]_0^\delta$$

$$\left\{\because \int \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy = \frac{\frac{\pi y}{2\delta}\times\frac{1}{2}}{\frac{\pi}{2\delta}} - \frac{\sin 2\left(\frac{\pi}{2}\frac{y}{\delta}\right)}{4\times\frac{\pi}{2\delta}}\right\}$$

$$\because \quad \frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\left(\frac{-\cos\frac{\pi}{2}\frac{\delta}{\delta}}{\frac{\pi}{2\delta}} + \frac{\cos\frac{\pi}{2}\times\frac{0}{\delta}}{\frac{\pi}{2\delta}}\right) - \left[\frac{\frac{\pi}{2}\frac{\delta}{\delta}\times\frac{1}{2}}{\frac{\pi}{2\delta}} - 0\right]\right]$$

$$= \frac{\partial}{\partial x}\left[\left(0 + \frac{1}{\frac{\pi}{2\delta}}\right) - \frac{\left(\frac{\pi}{4}\right)}{\frac{\pi}{2\delta}}\right] = \frac{\partial}{\partial x}\left[\frac{2\delta}{\pi} - \frac{\pi}{4} \times \frac{2\delta}{\pi}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{2\delta}{\pi} - \frac{\delta}{2}\right] = \frac{\partial}{\partial x}\left[\frac{4-\pi}{2\pi}\right]\delta = \left(\frac{4-\pi}{2\pi}\right)\frac{\partial\delta}{\partial x}$$

$\therefore$
$$\tau_0 = \left(\frac{4-\pi}{2\pi}\right)\rho U^2 \frac{\partial\delta}{\partial x}$$

τ_0 is also equal $= \mu \left(\frac{du}{dy}\right)_{\text{at } y=0}$

But
$$u = U \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

$\therefore$
$$\left(\frac{du}{dy}\right) = U \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) \times \frac{\pi}{2\delta}$$

$$\left(\frac{du}{dy}\right)_{y=0} = U \times \frac{\pi}{2\delta}\cos\left(\frac{\pi}{2} \times \frac{0}{\delta}\right) = \frac{U\pi}{2\delta}$$

$$\therefore \qquad \tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U \pi}{2\delta} \qquad \text{...(13.31)}$$

Equating the two values τ_0 given by equations (13.30) and (13.31)

$$\left(\frac{4-\pi}{2\pi}\right)\rho U^2 \frac{\partial \delta}{\partial x} = \frac{\mu U \pi}{2\delta} \quad \text{or} \quad \delta \partial \delta = \frac{\mu U \pi}{2} \times \frac{2\pi}{4-\pi} \times \frac{1}{\rho U^2} \, \partial x$$

$$\therefore \qquad \delta \partial \delta = \frac{\pi^2}{(4-\pi)} \frac{\mu U}{\rho U^2} \, . \, \partial x = 11.4975 \frac{\mu}{\rho U} \partial x$$

Integrating, we get $\dfrac{\delta^2}{2} = 11.4975 \dfrac{\mu}{\rho U} x + C$

At $x = 0$, $\delta = 0$ and hence $C = 0$

$$\therefore \qquad \frac{\delta^2}{2} = 11.4975 \frac{\mu}{\rho U} \, x$$

$$\therefore \qquad \delta = \sqrt{2 \times 11.4975 \frac{\mu}{\rho U} x} = 4.795 \sqrt{\frac{\mu}{\rho U} \, x}$$

$$= 4.795 \sqrt{\frac{\mu}{\rho U x}} = 4.795 \sqrt{\frac{\mu}{\rho U x}} \times x$$

$$= \frac{4.795 \, x}{\sqrt{R_{e_x}}} \qquad \text{...(13.32)}$$

(*ii*) **Shear Stress (τ_0)**

From equation (13.31),
$$\tau_0 = \frac{\mu U \pi}{2\delta} = \frac{\mu U \pi}{\dfrac{2 \times 4.795\, x}{\sqrt{R_{e_x}}}} = \frac{\mu U \pi \sqrt{R_{e_x}}}{2 \times 4.795\, x}$$

$$= \frac{\pi}{2 \times 4.795} \frac{\mu U}{x} \sqrt{R_{e_x}} = 0.327 \frac{\mu U}{x} \sqrt{R_{e_x}} .$$

(*iii*) **Drag force (F_D)** on one side of the plate is given by equation (13.12)

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.327 \frac{\mu U}{x} \sqrt{R_{e_x}} \times b \times dx = 0.327\, \mu U \times b \int_0^L \frac{1}{x} \sqrt{\frac{\rho U x}{\mu}}\, dx$$

$$= 0.327\, \mu U \times b \times \sqrt{\frac{\rho U}{\mu}} \int_0^L x^{-1/2}\, dx = 0.327\, \mu U \times b \times \sqrt{\frac{\rho U}{\mu}} \left[\frac{x^{1/2}}{\frac{1}{2}} \right]_0^L$$

$$= 0.327 \times 2 \times \mu U \times b \sqrt{\frac{\rho U}{\mu}} \times \sqrt{L}$$

$$= 0.655 \times \mu U \times b \times \sqrt{\frac{\rho U L}{\mu}} \qquad \text{...(13.33)}$$

(*iv*) **Co-efficient of drag, C_D** is given by equation (13.14),

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}, \text{ where } A = b \times L$$

$$\therefore \quad C_D = \frac{0.655 \times \mu U \times b \times \sqrt{\frac{\rho U L}{\mu}}}{\frac{1}{2}\rho U^2 \times b \times L} = 0.655 \times 2 \times \frac{\mu}{\rho U L} \times \sqrt{\frac{\rho U L}{\mu}}$$

$$= 1.31 \times \frac{1}{\sqrt{\frac{\rho U L}{\mu}}} = \frac{1.31}{\sqrt{R_{e_L}}} \qquad \ldots(13.34)$$

Problem 13.8 *For the velocity profile in laminar boundary layer as,*

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

find the thickness of the boundary layer and the shear stress 1.5 m from the leading edge of a plate. The plate is 2 m long and 1.4 m wide and is placed in water which is moving with a velocity of 200 mm per second. Find the total drag force on the plate if μ for water = .01 poise.

Solution. Given :

Velocity profile is $$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Distance of x from leading edge, $x = 1.5$ m

Length of plate, $L = 2$ m

Width of plate, $b = 1.4$ m

Velocity of plate, $U = 200$ mm/s $= 0.2$ m/s

Viscosity of water, $\mu = 0.01 \text{ poise} = \frac{0.01}{10} = 0.001 \text{ Ns/m}^2$

For the given velocity profile, thickness of boundary layer is given by equation (13.22) as

$$\delta = \frac{4.64\, x}{\sqrt{R_{e_x}}}$$

$$\left[\text{Here } R_{e_x} = \frac{\rho U x}{\mu} = 1000 \times \frac{0.2 \times 1.5}{0.001} = 300000\right]$$

$$\delta = \frac{4.64 \times 1.5}{\sqrt{300000}} = 0.0127 \text{ m} = \mathbf{12.7 \text{ mm. Ans.}}$$

Shear stress (τ_0) is given by $\tau_0 = 0.323 \frac{\mu U}{x} \sqrt{R_{e_x}}$

$$= 0.323 \times 0.001 \times \frac{0.2}{1.5} \times \sqrt{300000} = \mathbf{0.0235 \text{ N/m}^2\text{. Ans.}}$$

Drag Force (F_D) on one side of the plate is given by (13.23) as

$$F_D = 0.646\, \mu U \sqrt{\frac{\rho U L}{\mu}} \times b$$

$$= 0.646 \times 0.001 \times 0.2 \times \sqrt{1000 \times \frac{0.2 \times 2.0}{0.001}} \times 1.4$$

$$= .646 \times 0.001 \times 0.2 \times \sqrt{400000} \times 1.4 = 0.1138 \text{ N}$$

$\therefore$ Total drag force = Drag force on both sides of the plate

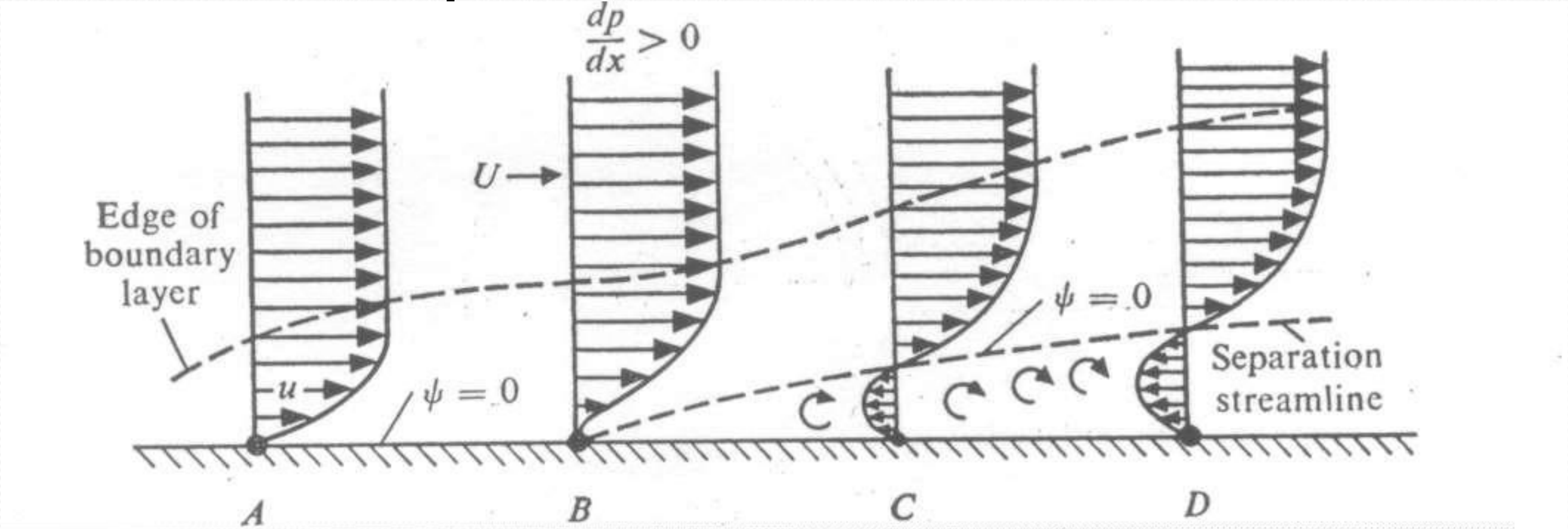
Boundary Layer Separation

- The increasing downstream pressure slows down the wall flow and can make it go backward-flow separation.

- *dp/dx* >0 adverse pressure gradient, flow separation may occur.

- *dp/dx* < 0 favourable gradient, flow is very resistant to separation.

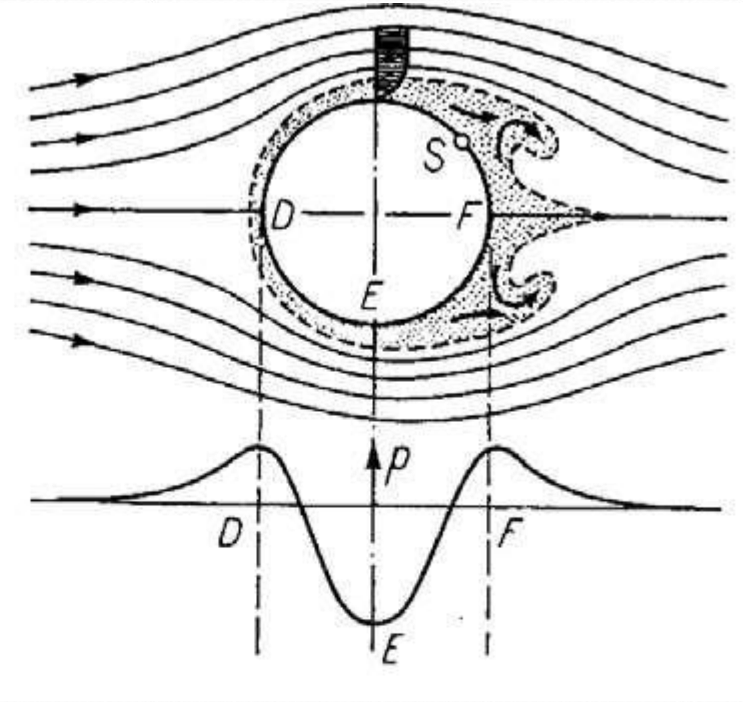
BL Separation Condition



❖Due to backflow close to the wall, a strong thickening of the BL takes place and BL mass is transported away into the outer flow.

❖At the point of separation, the streamlines leave the wall at a certain angle.

Separation Of BL At A Circular Cylinder



Separation of the boundary layer and vortex formation a circular cylinder (schematic). S=separation point

- D to E, pressure drop, pressure is transformed into kinetic energy.

- From E to F, kinetic energy is transformed into pressure.

- A fluid particle directly at the wall in the boundary layer is also acted upon by the same pressure distribution as in the outer flow(inviscid).

- Due to the strong friction forces in the BL, a BL particle loses so much of its kinetic energy that is cannot manage to get over the “pressure gradient” from E to F.

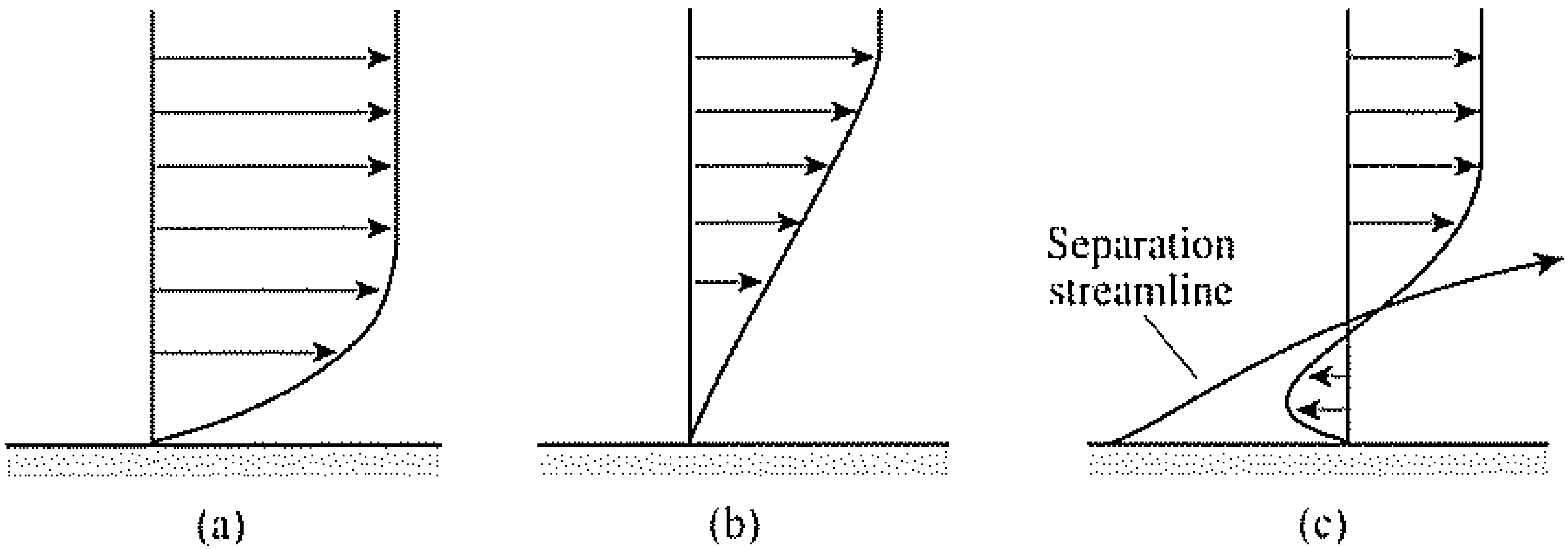
- The following figure shows the time sequence of this process:

- Reversed motion begun at the trailing edge.

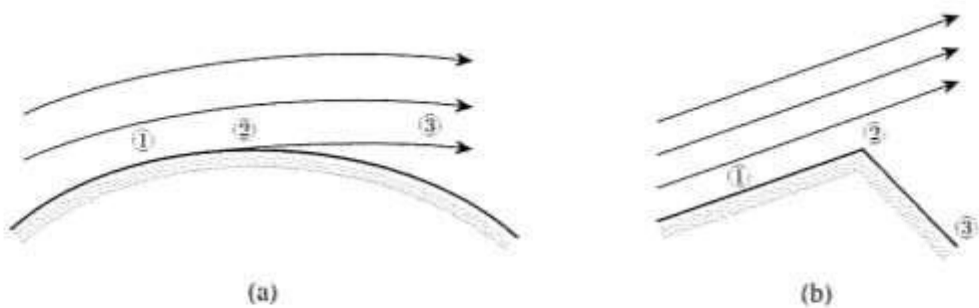
- Boundary layer has been thickened, and start of the reversed motion has moved forward considerably.

- And d. a large vortex formed from the backflow and then soon separates from the body.

EXAMPLE OF FLOW SEPARATION



- Velocity profiles in a boundary layer subjected to a pressure rise
  - (a) start of pressure rise
  - (b) after a small pressure rise
  - (c) after separation
- Flow separation from a surface
  - (a) smooth body
  - (b) salient edge



1. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is negative ... the flow has separated.

2. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ = 0 ... the flow is on the verge of separation.

3. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is positive ... the flow will not separate or flow will remain attached with the surface.

Problem 13.18 *For the following velocity profiles, determine whether the flow has separated or on the verge of separation or will attach with the surface :*

(*i*) $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$, (*ii*) $\frac{u}{U} = 2\left(\frac{y}{\delta}\right)^2 - \left(\frac{y}{\delta}\right)^3$,

(*iii*) $\frac{u}{U} = -2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2$.

Solution. Given :

1st Velocity Profile

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad \text{or} \quad u = \frac{3U}{2}\left(\frac{y}{\delta}\right) - \frac{U}{2}\left(\frac{y}{\delta}\right)^3$$

Differentiating w.r.t. y, the above equation becomes,

$$\frac{\partial u}{\partial y} = \frac{3U}{2} \times \frac{1}{\delta} - \frac{U}{2} \times 3\left(\frac{y}{\delta}\right)^2 \times \frac{1}{\delta}$$

At $y = 0$, $\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{3U}{2\delta} - \frac{3U}{2}\left(\frac{0}{\delta}\right)^2 \times \frac{1}{\delta} = \frac{3U}{2\delta}$.

As $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is positive. Hence flow will not separate or flow will remain attached with the surface.

2nd Velocity Profile

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right)^2 - \left(\frac{y}{\delta}\right)^3$$

$$\therefore \qquad u = 2U\left(\frac{y}{\delta}\right)^2 - U\left(\frac{y}{\delta}\right)^3$$

$$\therefore \qquad \frac{\partial u}{\partial y} = 2U \times 2\left(\frac{y}{\delta}\right) \times \frac{1}{\delta} - U \times 3\left(\frac{y}{\delta}\right)^2 \times \frac{1}{\delta}$$

at $y = 0$, $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 2U \times 2\left(\frac{0}{\delta}\right) \times \frac{1}{\delta} - U \times 3\left(\frac{0}{\delta}\right)^2 \times \frac{1}{\delta} = 0$

As $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$, the flow is on the verge of separation. **Ans.**

3rd Velocity Profile

$$\frac{u}{U} = -2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2$$

$$\therefore \qquad u = -2U\left(\frac{y}{\delta}\right) + U\left(\frac{y}{\delta}\right)^2$$

$$\therefore \qquad \frac{\partial u}{\partial y} = -2U\left(\frac{1}{\delta}\right) + 2U\left(\frac{y}{\delta}\right) \times \frac{1}{\delta}$$

at $y = 0$, $\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\frac{2U}{\delta} + 2U\left(\frac{0}{\delta}\right) \times \frac{1}{\delta} = -\frac{2U}{\delta}$

As $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is negative the flow has separated. **Ans.**

Dimensionless numbers

- It is almost impossible to test models at physical sizes similar to the actual full-scale prototypes → try testing A380 in a wind tunnel!
- Thankfully, research has shown that if similarities are achieved between model and prototype, scaling of data is possible
  - Three kinds of similarity
    - *Geometric similarity*
    - *Dynamic similarity*
    - *Kinematic similarity*
- Realistically, not all three similarities can be achieved simultaneously. Sometimes, even impossible
- Similarities are achieved by comparing dimensionless numbers
  - Reynolds number
  - Mach number etc

Dimensions

QUANTITY	DEFINING EQUATION	DIMENSIONS, MLT SYSTEM
Geometrical		
Angle	Arc/Radius (a ratio)	$[M^0 L^0 T^0]$
Length	(Including all linear measurement)	$[L]$ ← Basic dimension
Area	Length × Length	$[L^2]$
Volume	Area × Length	$[L^3]$
Kinematic		
Time	–	$[T]$ ← Basic dimension
Velocity, linear	Distance/Time	$[LT^{-1}]$
Acceleration, linear	Linear velocity/Time	$[LT^{-2}]$
Velocity, angular	Angle/Time	$[T^{-1}]$
Acceleration, angular	Angular velocity/Time	$[T^{-2}]$
Volume rate of discharge	Volume/Time	$[L^3T^{-1}]$
Dynamic		
Mass	Force/Acceleration	$[M]$ ← Basic dimension
Force	Mass × Acceleration	$[MLT^{-2}]$
Weight	Force	$[MLT^{-2}]$
Mass density	Mass/Volume	$[ML^{-3}]$
Work, energy	Force × Distance	$[ML^2T^{-2}]$
Power	Work/Time	$[ML^2T^{-3}]$
Moment of a force	Force × Distance	$[ML^2T^{-2}]$
Viscosity, dynamic	Shear stress/Velocity gradient	$[ML^{-1}T^{-1}]$
Viscosity, kinematic	Dynamic viscosity/ Mass density	$[L^2T^{-1}]$

▶ 12.3 DIMENSIONAL HOMOGENEITY

Dimensional homogeneity means the dimensions of each terms in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation. The powers of fundamental dimensions (*i.e.*, L, M, T) on both sides of the equation will be identical for a dimensionally homogeneous equation. Such equations are independent of the system of units.

Let us consider the equation, $V = \sqrt{2gH}$

Dimension of L.H.S. $= V = \dfrac{L}{T} = LT^{-1}$

Dimension of R.H.S. $= \sqrt{2gH} = \sqrt{\dfrac{L}{T^2} \times L} = \sqrt{\dfrac{L^2}{T^2}} = \dfrac{L}{T} = LT^{-1}$

Dimension of L.H.S. = Dimension of R.H.S. $= LT^{-1}$

$\therefore$ Equation $V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

The dimensional formula of Acceleration Due To Gravity is given by,

$$[M^0 L^1 T^{-2}]$$

Where,

- M = Mass
- L = Length
- T = Time

Derivation

Since, Force = Mass × Acceleration due to gravity

∴ Acceleration due to gravity (g) = Force × $[\text{Mass}]^{-1}$ . . . . . (1)

⇒ Dimensional formula of the mass = $[M^1 L^0 T^0]$ . . . . . (2)

Also, the dimensions of Force = $[M^1 L^1 T^{-2}]$ . . . . (3)

On substituting equation (2) and (3) in equation (1) we get,

Acceleration due to gravity = Force × $[\text{Mass}]^{-1}$

Or, g = $[M^1 L^1 T^{-2}] \times [M^1 L^0 T^0]^{-1} = [M^0 L^1 T^{-2}]$.

Therefore, acceleration due to gravity is dimensionally represented as **$[M^0 L^1 T^{-2}]$.**

▶ 12.4 METHODS OF DIMENSIONAL ANALYSIS

If the number of variable involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods :

1. Rayleigh's method, and
2. Buckingham's π-theorem.

Problem 12.2 *The time period (t) of a pendulum depends upon the length (L) of the pendulum and acceleration due to gravity (g). Derive an expression for the time period.*

Solution. Time period t is a function of (*i*) L and (*ii*) g

$$\therefore \quad t = KL^a \cdot g^b, \text{ where } K \text{ is a constant} \quad \text{...}(i)$$

Substituting the dimensions on both sides $T^1 = KL^a \cdot (LT^{-2})^b$

Equating the powers of M, L and T on both sides, we have

Power of T, $\quad 1 = -2b \quad \therefore \quad b = -\frac{1}{2}$

Power of L, $\quad 0 = a + b \quad \therefore \quad a = -b = -\left(-\frac{1}{2}\right) = \frac{1}{2}$

Substituting the values of a and b in equation (*i*),

$$t = KL^{1/2} \cdot g^{-1/2} = K\sqrt{\frac{L}{g}}$$

The value of K is determined from experiments which is given as

$$K = 2\pi$$

$$\therefore \quad t = 2\pi\sqrt{\frac{\mathbf{L}}{\mathbf{g}}}. \textbf{ Ans.}$$

Problem 12.3 *Find an expression for the drag force on smooth sphere of diameter D, moving with a uniform velocity V in a fluid of density ρ and dynamic viscosity μ.*

Solution. Drag force F is a function of

(*i*) Diameter, D (*ii*) Velocity, V (*iii*) Density, ρ

(*iv*) Viscocity, μ

$$\therefore \quad F = KD^a \cdot V^b \cdot \rho^c \cdot \mu^d \quad \text{...(}i\text{)}$$

where K is non-dimensional factor.

Substituting the dimensions on both sides,

$$MLT^{-2} = KL^a \cdot (LT^{-1})^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d$$

Equating the powers of M, L and T on both sides,

Power of M, $\quad 1 = c + d$

Power of L, $\quad 1 = a + b - 3c - d$

Power of T, $\quad -2 = -b - d.$

There are four unknowns (a, b, c, d) but equations are three. Hence it is not possible to find the values of a, b, c and d. But three of them can be expressed in terms of fourth variable which is most important. Here viscosity is having a vital role and hence a, b, c are expressed in terms of d which is the power to viscosity.

$$\therefore \quad c = 1 - d$$

$$b = 2 - d$$

$$a = 1 - b + 3c + d = 1 - 2 + d + 3(1 - d) + d$$

$$= 1 - 2 + d + 3 - 3d + d = 2 - d$$

Substituting these values of a, b and c in (*i*), we get

$$F = KD^{2-d} \cdot V^{2-d} \cdot \rho^{1-d} \cdot \mu^d$$

$$= KD^2V^2\rho\,(D^{-d} \cdot V^{-d} \cdot \rho^{-d} \cdot \mu^d) = K\rho D^2V^2 \left(\frac{\mu}{\rho VD}\right)^d$$

$$= \mathbf{K\rho D^2V^2\phi\left(\frac{\mu}{\rho VD}\right)} \text{. Ans.}$$

Problem 12.6 *The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft l, velocity V, air viscosity μ, air density ρ and bulk modulus of air K. Express the functional relationship between these variables and the resisting force.*

Solution. The resisting force R depends upon

(*i*) density, l, (*ii*) velocity, V,

(*iii*) viscosity, μ, (*iv*) density, ρ,

(*v*) Bulk modulus, K.

$$\therefore \quad R = Al^a \cdot V^b \cdot \mu^c \cdot \rho^d \cdot K^e \quad \text{...}(i)$$

where A is the non-dimensional constant.

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = AL^a \cdot (LT^{-1})^b \cdot (ML^{-1}T^{-1})^c \cdot (ML^{-3})^d \cdot (ML^{-1}T^{-2})^e$$

Equating the powers of M, L, T on both sides,

Power of M, $1 = c + d + e$

Power of L, $1 = a + b - c - 3d - e$

Power of T, $-2 = -b - c - 2e.$

There are five unknowns but equations are only three. Expressing the three unknowns in terms of two unknowns (μ and K).

$\therefore$ Express the values of a, b and d in terms of c and e.

Solving,

$$d = 1 - c - e$$

$$b = 2 - c - 2e$$

$$a = 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3\,(1 - c - e) + e$$

$$= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c.$$

Substituting these values in (*i*), we get

$$R = A\, l^{2-c} \cdot V^{2-c-2e} \cdot \mu^c \cdot \rho^{1-c-e} \cdot K^e$$

$$= A\, l^2 \cdot V^2 \cdot \rho(l^{-c}\, V^{-c}\, \mu^c\, \rho^{-c}) \cdot (V^{-2e} \cdot \rho^{-e} \cdot K^e)$$

$$= A\, l^2 V^2 \rho \left(\frac{\mu}{\rho VL}\right)^c \cdot \left(\frac{K}{\rho V^2}\right)^e$$

$$= \mathbf{A\rho l^2 V^2\ \phi \left[\left(\frac{\mu}{\rho VL}\right) \cdot \left(\frac{K}{\rho V^2}\right)\right]}. \textbf{ Ans.}$$

Dimensionless numbers

- ❖ Dimensionless numbers are obtained by
  - ✓ *Buckingham Π theorem* (simple but can be tedious)
  - ✓ *Inspection of variables* (not so simple but fast)

- ❖ Buckingham Π theorem
  - ✓ "If an equation involving *k* number of variables is dimensionally homogeneous, it can be reduced to a relationship among *k*-*r* number of independent dimensionless products, where *r* is the minimum number of reference dimensions required to describe the variables."

- ❖ Involves rewriting the variables in their basic dimensions, M, L and T and rearrange them such that they equate to a dimensionless term

- ❖ Recall: M – mass, L – length, T – time

- ❖ Dimensionless term → $M^0L^0T^0$

Dimensionless numbers

- Procedures
  1. List down all critical variables involved in the flow problem.
  2. Express each of the variables in basic dimensions.
  3. Determine the number of independent dimensionless products i.e. $\prod$ terms.
  4. Select a number of repeating variables, such that the number required is equal to the number of reference dimensions.
  5. Form a $\prod$ term by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.
  6. Repeat the previous step for all the remaining non-repeating variables.
  7. Ensure all the resulting $\prod$ terms are dimensionless.
  8. Express the final form of the "$\prod$ function" in terms of the $\prod$ terms.

Solution. **Step 1.** The resisting force R depends upon (*i*) l, (*ii*) V, (*iii*) μ, (*iv*) ρ and (*v*) K. Hence R is a function of l, V, μ, ρ and K. Mathematically,

$$R = f(l, V, \mu, \rho, K) \qquad ...(i)$$

or it can be written as $f_1 (R, l, V, \mu, \rho, K) = 0$ ...(*ii*)

$\therefore$ Total number of variables, $n = 6$.

Number of fundamental dimensions, $m = 3$.

[m is obtained by writing dimensions of each variables as $R = MLT^{-2}$, $V = LT^{-1}$, $\mu = ML^{-1}T^{-1}$, $\rho = ML^{-3}$, $K = ML^{-1}T^{-2}$. Thus as fundamental dimensions in the problem are M, L, T and hence $m = 3$.]

Number of dimensionless π-terms $= n - m = 6 - 3 = 3$.

Thus three π-terms say π_1, π_2 and π_3 are formed. Hence equation (*ii*) is written as

$$f_1 (\pi_1, \pi_2, \pi_3) = 0. \qquad ...(iii)$$

Step 2. Each π term $= m + 1$ variables, where m is equal to 3 and also called repeating variables. Out of six variables R, l, V, μ, ρ and K, three variables are to be selected as repeating variable. R is a dependent variable and should not be selected as a repeating variable. Out of the five remaining variables, one variable should have geometric property, the second variable should have flow property and third one fluid property. These requirements are fulfilled by selecting l, V and ρ as repeating variables. The repeating variables themselves should not form a dimensionless term and should have themselves fundamental dimensions equal to m, *i.e.*, 3 here. Dimensions of l, V and ρ are L, LT^{-1}, ML^{-3} and hence the three fundamental dimensions exist in l, V and ρ and they themselves do not form dimensionless group.

Step 3. Each π-term is written as according to equation (12.4)

$$\left.\begin{aligned} \pi_1 &= l^{a_1} . V^{b_1} . \rho^{c_1} . R \\ \pi_2 &= l^{a_2} . V^{b_2} . \rho^{c_2} . \mu \\ \pi_3 &= l^{a_3} . V^{b_3} . \rho^{c_3} . K \end{aligned}\right\} \quad \text{...}(iv)$$

Step 4. Each π-term is solved by the principle of dimensional homogeneity. For the first π-term, we have

$$\pi_1 = M^0L^0T^0 = L^{a_1} . (LT^{-1})^{b_1} . (ML^{-3})^{c_1} . MLT^{-2}.$$

Equating the powers of M, L, T on both sides, we get

Power of M, $\quad 0 = c_1 + 1 \quad \therefore \quad c_1 = -1$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1,$

$\therefore \quad a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Power of T, $\quad 0 = -b_1 - 2 \quad \therefore \quad b_1 = -2$

Substituting the values of a_1, b_1 and c_1 in equation (iv),

$$\pi_1 = l^{-2} . V^{-2} . \rho^{-1} . R$$

or

$$\pi_1 = \frac{R}{l^2V^2\rho} = \frac{R}{\rho l^2 V^2} \quad \text{...}(v)$$

Similarly for the 2nd π-term, we get $\pi_2 = M^0L^0T^0 = L^{a_2} \,.\, (LT^{-1})^{b_2} \,.\, (ML^{-3})^{c_2} \,.\, ML^{-1}T^{-1}$.

Equating the powers of M, L, T on both sides

Power of M, $\quad 0 = c_2 + 1, \quad \therefore \quad c_2 = -1$

Power of L, $\quad 0 = a_2 + b_2 - 3c_2 - 1,$

$$a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$$

Power of T, $\quad 0 = -b_2 - 1, \quad \therefore \quad b_2 = -1$

Substituting the values of a_2, b_2 and c_2 in π_2 of (*iv*)

$$\pi_2 = l^{-1} \,.\, V^{-1} \,.\, \rho^{-1} \,.\, \mu = \frac{\mu}{lV\rho}.$$

3rd π-term

$$\pi_3 = l^{a_3} \,.\, V^{b_3} \,.\, \rho^{c_3} \,.\, K$$

or

$$M^0L^0T^0 = L^{a_3} \,.\, (LT^{-3})^{b_3} \,.\, (ML^{-3})^{a_3} \,.\, ML^{-1}T^{-2}$$

Equating the powers of M, L, T on both sides, we have

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \quad c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \quad a_3 = -b_3 + 3c_3 + 1 = 2 - 3 + 1 = 0$

Power of T, $\quad 0 = -b_3 - 2, \quad \therefore \quad b_3 = -2$

Substituting the values of a_3, b_3 and c_3 in π_3 term

$$\pi_3 = l^0 \cdot V^{-2} \,.\, \rho^{-1} \,.\, K = \frac{K}{V^2\rho}.$$

Step 5. Substituting the values of π_1, π_2 and π_3 in equation (*iii*), we get

$$f_1\left(\frac{R}{\rho l^2V^2}, \frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right) = 0 \quad \text{or} \quad \frac{R}{\rho l^2V^2} = \phi\left[\frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right]$$

or

$$R = \rho \mathbf{l}^2\mathbf{V}^2\phi\left[\frac{\mu}{\mathbf{lV}\rho}, \frac{\mathbf{K}}{\mathbf{V}^2\rho}\right]. \textbf{ Ans.}$$

Example

- It is known that the fluid pressure drop per unit length of the pipe, Δp_l, has to be dependent on the fluid velocity, U, pipe diameter, D, fluid density, ρ, and fluid dynamic viscosity, μ. Base on this information, derive the dimensionless terms.

[Solution]

Dependent variables

Independent variables

We can write: $$\Delta p_l = f\left(U, D, \rho, \mu\right)$$

Express all variables in basic dimensions:

$$\Delta p_l \equiv ML^{-2}T^{-2} \qquad D \equiv L \qquad \rho \equiv ML^{-3} \qquad \mu \equiv ML^{-1}T^{-1} \qquad U \equiv LT^{-1}$$

Number of basic dimensions used – 3

Number of variables – 5 → Number of Π terms: 5 – 3 = 2

Now, pick repeating variables to form the two Π terms, bearing in mind that:

- Do not use the dependent variable
- Select the variables with the simplest basic dimensions

Example

Using D, U and ρ as the repeating variables, combine them with the dependent variable to get the two Π terms:

$$\Pi_1 = \Delta\rho_l D^a U^b \rho^c$$

$$\Pi_2 = \mu D^d U^e \rho^f$$

Since both Π terms are dimensionless:

$$\Pi_1 = \left(ML^{-2}T^{-2}\right)\left(L\right)^a \left(LT^{-1}\right)_b \left(ML^{-3}\right)_c = M^0 L^0 T^0$$

Equating the exponents on the left and right hand side:

$$1 + c = 0$$

$$-2 + a + b - 3c = 0$$

$$-2 - b = 0$$

It can be worked out that a = 1, b = -2 and c = -1:

$$\Rightarrow \Pi_1 = \frac{\Delta\rho_l D}{\rho U^2}$$

Example

Repeating for the second Π term:

$$\Pi_2 = \left(ML^{-1}T^{-1}\right)(L)^d \left(LT^{-1}\right)_e \left(ML^{-3}\right)_f = M^0L^0T^0$$

$$1 + f = 0$$

$$-1 + d + e - 3f = 0$$

$$-1 - e = 0$$

It can be worked out that d = -1, e = -1 and f = -1:

$$\Rightarrow \Pi_2 = \frac{\mu}{\rho UD}$$

Note that reciprocal forms of the Π terms as valid → dimensionless anyway

Equally valid to write:

$$\Pi_1 = \frac{\rho U^2}{\Delta \rho D} \qquad \Pi_2 = \frac{\rho UD}{\mu}$$

→ Does this look familiar to you?

Problem 12.8 *(a) State Buckingham's π-theorem.*

(b) The efficiency η of a fan depends on density ρ, dynamic viscosity μ of the fluid, angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.

Solution. (*a*) Statement of Buckingham's π-theorem is given in Article 12.4.2.

(*b*) Given : η is a function of ρ, μ, ω, D and Q

$$\therefore \qquad \eta = f(\rho, \mu, \omega, D, Q) \quad \text{or} \quad f_1(\eta, \rho, \mu, \omega, D, Q) = 0 \qquad \ldots(i)$$

Hence total number of variables, $n = 6$.

The value of m, *i.e.*, number of fundamental dimensions for the problem is obtained by writing dimensions of each variable. Dimensions of each variable are

$$\eta = \text{Dimensionless},\ \rho = ML^{-3},\ \mu = ML^{-1}T^{-1},\ \omega = T^{-1},\ D = L \text{ and } Q = L^3T^{-1}$$

$$\therefore \qquad m = 3$$

Number of π-terms $= n - m = 6 - 3 = 3$

Equation (i) is written as $f_1(\pi_1, \pi_2, \pi_3) = 0$ ...(ii)

Each π-term contains $m + 1$ variables, where m is equal to three and is also repeating variable.

Choosing D, ω and ρ as repeating variables, we have

$$\pi_1 = D^{a_1} . \omega^{b_1} . \rho^{c_1} . \eta$$
$$\pi_2 = D^{a_2} . \omega^{b_2} . \rho^{c_2} . \mu$$
$$\pi_3 = D^{a_3} . \omega^{b_3} . \rho^{c_3} . Q$$

First π-term $\pi_1 = D^{a_1} . \omega^{b_1} . \rho^{c_1} . \eta$

Substituting dimensions on both sides of π_1,

$$M^0L^0T^0 = L^{a_1} . (T^{-1})^{b_1} . (ML^{-3})^{c_1} . M^0L^0T^0$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_1 + 0$, $\therefore$ $c_1 = 0$

Power of L, $0 = a_1 + 0$, $\therefore$ $a_1 = 0$

Power of T, $0 = -b_1 + 0$, $\therefore$ $b_1 = 0$

Substituting the values of a_1, b_1 and c_1 in π_1, we get

$$\pi_1 = D^0\omega^0\rho^0 . \eta = \eta$$

[If a variable is dimensionless, it itself is a π-term. Here the variable η is a dimensionless and hence η is a π-term. As it exists in first π-term and hence $\pi_1 = \eta$. Then there is no need of equating the powers. Directly the value can be obtained.]

Second π-term $\pi_2 = D^{a_2} . \omega^{b_2} . \rho^{c_2} . \mu$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_2} . (T^{-1})^{b_2} . (ML^{-3})^{c_2} . ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_2 + 1$, $\therefore$ $c_2 = -1$

Power of L, $0 = a_2 - 3c_2 - 1$, $\therefore$ $a_2 = 3c_2 + 1 = -3 + 1 = -2$

Power of T, $0 = -b_2 - 1$, $\therefore$ $b_2 = -1$

Substituting the values of a_2, b_2 and c_2 in π_2,

$$\pi_2 = D^{-2} \cdot \omega^{-1} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{D^2\omega\rho}$$

Third π-term $\pi_3 = D^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot Q$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_3} \cdot (T^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot L^3T^{-1}$$

Equating the powers of M, L and T on both sides

Power of M, $0 = c_3$, $\therefore$ $c_3 = 0$

Power of L, $0 = a_3 - 3c_3 + 3$, $\therefore$ $a_3 = 3c_3 - 3 = -3$

Power of T, $0 = -b_{3\,-1}$, $\therefore$ $b_3 = -1$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^{-3} \cdot \omega^{-1} \cdot \rho^{0} \cdot Q = \frac{Q}{D^2\omega}$$

Substituting the values of π_1, π_2 and π_3 in equation (*ii*)

$$f_1\left(\eta, \frac{\mu}{D^2\omega\rho}, \frac{Q}{D^2\omega}\right) = 0 \text{ or } \eta = \phi\left[\frac{\mu}{\mathbf{D^2\omega\rho}}, \frac{\mathbf{Q}}{\mathbf{D^2\omega}}\right]. \textbf{ Ans.}$$

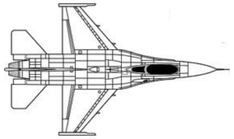
Dimensionless numbers

Name	Formula	Type of force ratio	Applications
Reynolds number, Re	$\mathrm{Re} = \frac{\rho U L}{\mu}$	$\frac{\text{inertia force}}{\text{viscous force}}$	Almost all fluid problems
Froude number, Fr	$Fr = \frac{U}{gL}$	$\frac{\text{inertia force}}{\text{gravitational force}}$	Free surface flows
Euler number, Eu	$Eu = \frac{p}{\rho V^2}$	$\frac{\text{pressure force}}{\text{inertia force}}$	Flows with pressure differentials
Mach number, Ma	$Ma = \frac{U}{a}$	$\frac{\text{inertia force}}{\text{compressibility force}}$	Compressible flows
Strouhal number, St	$St = \frac{fL}{U}$	$\frac{\text{local inertia force}}{\text{convective inertia force}}$	Fluctuating or oscillating flows
Weber number	$We = \frac{\rho U^2 L}{\sigma}$	$\frac{\text{inertia force}}{\text{surface tension force}}$	Flows with surface tensions

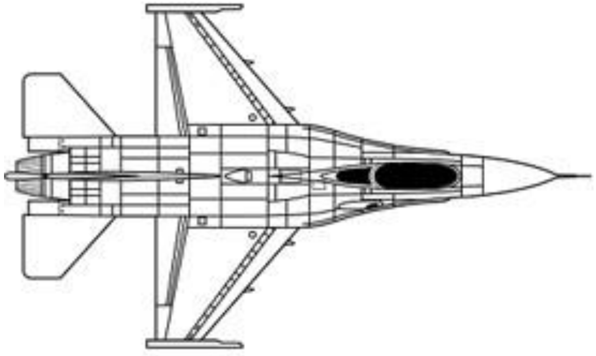
Similarity (similitude)

- Three similarity types in fluid flow studies
  - *Geometric similarity*
  - *Dynamic similarity*
  - *Kinematic similarity*

- **Geometric similarity**
  - Model and prototype share the same geometry
  - Length ratios on model same as length ratios on prototype
  - Difficult when surface roughness is important (why?)



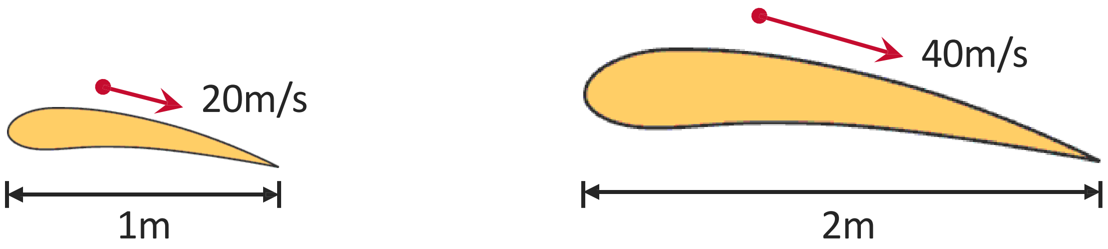
Model



Prototype

Similarity (similitude)

- **Kinematic similarity**
  - Similarity of motion, not geometry
  - Geometric similarity is implied + temporal similarity
  - Velocities and accelerations must be similar between model and prototype



  - Kinematic similarity → flow patterns will be geometrically similar (just bigger or smaller)
  - More difficult to achieve than geometric similarity but not impossible

Similarity (similitude)

- **Dynamic similarity**
  - Similarity of forces (i.e. lift, drag, gravitational etc) → very important
  - Geometric similarity must be satisfied, kinematic similarity usually satisfied
  - Associated with dimensionless numbers (recall that they are ratios of forces)

- For dynamic similarity

Dimensionless number of model = Dimensionless number of prototype

- For example, $Re_m = Re_p$ $St_m = St_p$ $Ma_m = Ma_p$

- So, is model scaling and testing really this simple, just matching all the important dimensionless numbers?

Similarity (similitude)

- Suppose we want to match Reynolds number for a subsonic aircraft model with its real-world prototype during wind tunnel testing:

$$\Rightarrow \frac{\rho_m u_m l_m}{\mu_m} = \frac{\rho_p u_p l_p}{\mu_p}$$

$$\Rightarrow u_m = u_p \left(\frac{l_p}{l_m}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{\mu_m}{\mu_p}\right)$$

1/32th scale F-16 model

- Typically, l_p/l_m ratio is very large, and we can assume ρ and μ remain relatively constant

u_m → supersonic regime !

- Situation: prototype subsonic but model supersonic → shock waves → not realistic at all

- Solution: variable density wind tunnel → increase ρ_m

Example

- A ship is to be driven at 12 m/s in sea water. A model ship of 1/20th scale is to be tested to determine the resistance encountered by the actual ship. Determine the velocity at which the model ship should be tested.

[Solution]

Similarity in free surface flows should be achieved → similar Froude number

$$\frac{U_m}{\sqrt{gl_m}} = \frac{U_p}{\sqrt{gl_p}}$$

$$\Rightarrow U_m = U_p \sqrt{\frac{l_m}{l_p}}$$

$$= (12)\sqrt{\frac{1}{20}} = 2.68 \text{ m/s}$$

→ Thus, the model ship should be tested at 2.68 m/s water speed.

Example

- A sphere experiences a drag force of 4.5 N when immersed in water moving with a velocity of 1.5 m/s. A second sphere twice the diameter is tested in a wind tunnel. If the two spheres are to have dynamic similarity, what should the air velocity in the wind tunnel be? Additionally, what will the drag force at this air velocity if the kinematic viscosity of air is 13 times that of water? Assume air density to be 1.28 kg/m^3.

[Solution]

For dynamic similarity, Reynolds numbers for both spheres should be similar in both air and water flows:

$$\frac{U_a D_a}{\nu_a} = \frac{U_w D_w}{\nu_w}$$

$$U_a = U_w \times \frac{D_w}{D_a} \times \frac{\nu_a}{\nu_w} = 1.5 \times \frac{1}{2} \times 13 = 9.75 \text{m/s}$$

→ Air velocity has to be 9.75 m/s for dynamic similarity.

Example

Next, drag coefficient should be the same for both spheres if the Reynolds numbers are similar:

$$C_D = \frac{F}{\frac{1}{2}\rho U^2 A}$$

Note: A is the area facing the flow (i.e. $A = \frac{1}{4}\pi D^2$)

$$(C_D)_a = (C_D)_w$$

$$F_a = F_w \times \frac{\rho_a}{\rho_w} \times \frac{U_a^2}{U_w^2} \times \frac{D_a^2}{D_w^2}$$

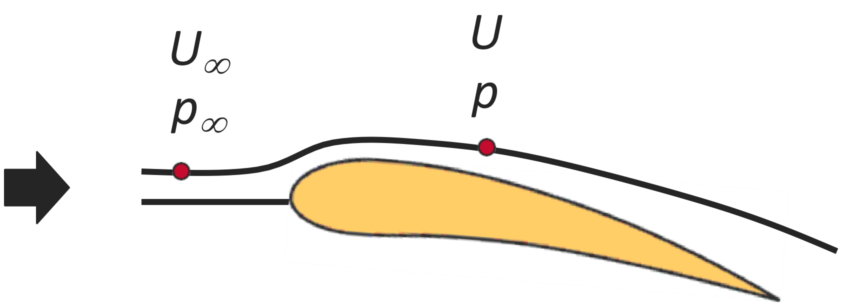
$$= 4.5 \times \frac{1.28}{1000} \times \left(\frac{9.75}{1.5}\right)^2 \times \left(\frac{2}{1}\right)^2 = 0.976\text{N}$$

→ Drag force at 9.7 5m/s will be 0.976 N.

Pressure, lift and drag coefficients

- Pressure coefficient, C_p, is very commonly used in aerodynamics and fluid mechanics

- Instead of using actual pressure values, non-dimensionalized pressure values are used → easier for scaling and comparisons



Pressure coefficient is defined as:

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}$$

$\leftarrow q_\infty$

- Writing the Bernoulli's equation for the above:

$$p_\infty + \tfrac{1}{2}\rho_\infty U_\infty^2 = p + \tfrac{1}{2}\rho_\infty U^2 \Rightarrow p - p_\infty = \tfrac{1}{2}\rho_\infty U_\infty^2\left(1 - \frac{U^2}{U_\infty^2}\right)$$

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = 1 - \frac{U^2}{U_\infty^2}$$

Pressure, lift and drag coefficients

- Lift and drag coefficients are used even more frequently and are defined as

$$C_L = \frac{F_L}{\frac{1}{2} A \rho_\infty U_\infty^2} \qquad C_D = \frac{F_D}{\frac{1}{2} A \rho_\infty U_\infty^2}$$

- Instead of plotting lift versus drag for an aerofoil/wing, C_L is plotted against C_D (or vice versa)
- We will encounter these non-dimensionalized terms very frequently later in the course

Model Analysis

1. The performance of the hydraulic structure or hydraulic machine can be easily predicted, in advance, from its model.

2. With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensionless parameters is obtained. This relationship helps in conducting tests on the model.

3. The merits of alternative designs can be predicted with the help of model testing. The most economical and safe design may be, finally, adopted.

4. The tests performed on the models can be utilized for obtaining, in advance, useful information about the performance of the prototypes only if a complete similarity exists between the model and the prototype.

▶ 12.7 TYPES OF FORCES ACTING IN MOVING FLUID

For the fluid flow problems, the forces acting on a fluid mass may be any one, or a combination of the several of the following forces :

1. Inertia force, F_i.	2. Viscous force, F_v.
3. Gravity force, F_g.	4. Pressure force, F_p.
5. Surface tension force, F_s.	6. Elastic force, F_e.

1. **Inertia Force** (F_i). It is equal to the product of mass and acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration. It is always existing in the fluid flow problems.

2. **Viscous Force** (F_v). It is equal to the product of shear stress (τ) due to viscosity and surface area of the flow. It is present in fluid flow problems where viscosity is having an important role to play.

3. **Gravity Force** (F_g). It is equal to the product of mass and acceleration due to gravity of the flowing fluid. It is present in case of open surface flow.

4. **Pressure Force** (F_p). It is equal to the product of pressure intensity and cross-sectional area of the flowing fluid. It is present in case of pipe-flow.

5. **Surface Tension Force** (F_s). It is equal to the product of surface tension and length of surface of the flowing fluid.

6. **Elastic Force** (F_e). It is equal to the product of elastic stress and area of the flowing fluid.

For a flowing fluid, the above-mentioned forces may not always be present. And also the forces, which are present in a fluid flow problem, are not of equal magnitude. There are always one or two forces which dominate the other forces. These dominating forces govern the flow of fluid.

1. Geometric Similarity. The geometric similarity is said to exist between the model and the prototype. The ratio of all corresponding linear dimension in the model and prototype are equal.

Let L_m = Length of model, b_m = Breadth of model,

D_m = Diameter of model, A_m = Area of model,

$\forall_m$ = Volume of model,

and $L_P, b_P, D_P, A_P, \forall_P$ = Corresponding values of the prototype.

For geometric similarity between model and prototype, we must have the relation,

$$\frac{L_P}{L_m} = \frac{b_P}{b_m} = \frac{D_P}{D_m} = L_r \qquad \text{...(12.6)}$$

where L_r is called the scale ratio.

For area's ratio and volume's ratio the relation should be as given below :

$$\frac{A_P}{A_m} = \frac{L_P \times b_P}{L_m \times b_m} = L_r \times L_r = L_r^2 \qquad \text{...(12.7)}$$

and

$$\frac{\forall_P}{\forall_m} = \left(\frac{L_P}{L_m}\right)^3 = \left(\frac{b_P}{b_m}\right)^3 = \left(\frac{D_P}{D_m}\right)^3 \qquad \text{...(12.8)}$$

2. Kinematic Similarity. Kinematic similarity means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist between the model and the prototype if the ratios of the velocity and acceleration at the corresponding points in the model and at the corresponding

For kinematic similarity, we must have

$$\frac{V_{P_1}}{V_{m_1}} = \frac{V_{P_2}}{V_{m_2}} = V_r$$

where V_r is the velocity ratio.

For acceleration, we must have $\dfrac{a_{P_1}}{a_{m_1}} = \dfrac{a_{P_2}}{a_{m_2}} = a_r$

3. Dynamic Similarity. Dynamic similarity means the similarity of forces between the model and prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratios of the corresponding forces acting at the corresponding points are equal. Also the directions of the corresponding forces at the corresponding points should be same.

Let $(F_i)_P$ = Inertia force at a point in prototype,

$(F_v)_P$ = Viscous force at the point in prototype,

$(F_g)_P$ = Gravity force at the point in prototype,

and $(F_i)_m$ $(F_v)_m$, $(F_g)_m$ = Corresponding values of forces at the corresponding point in model.

Then for dynamic similarity, we have

$$\frac{(F_i)_P}{(F_i)_m} = \frac{(F_v)_P}{(F_v)_m} = \frac{(F_g)_P}{(F_g)_m} \dots = F_r, \text{ where } F_r \text{ is the force ratio.}$$

Also the directions of the corresponding forces at the corresponding points in the model and prototype should be same.

12.9.1 Reynold's Model Law.

Reynold's model law is the law in which models are based on Reynold's number. Models based on Reynold's number includes :

(*i*) Pipe flow

(*ii*) Resistance experienced by sub-marines, airplanes, fully immersed bodies etc.

As defined earlier that Reynold number is the ratio of inertia force and viscous force, and hence fluid flow problems where viscous forces alone are predominent, the models are designed for dynamic similarity on Reynolds law, which states that the Reynold number for the model must be equal to the Reynold number for the prototype.

Let V_m = Velocity of fluid in model,

ρ_m = Density of fluid in model,

L_m = Length or linear dimension of the model,

μ_m = Viscosity or fluid in model,

and V_P, ρ_P, L_P and μ_P are the corresponding values of velocity, density, linear dimension and viscosity of fluid in prototype. Then according to Reynold's model law,

$$[R_e]_m = [R_e]_P \text{ or } \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_P V_P L_P}{\mu_P} \quad \text{...(12.17)}$$

12.9.2 Froude Model Law. Froude model law is the law in which the models are based on Froude number which means for dynamic similarity between the model and prototype, the Froude number for both of them should be equal. Froude model law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia. Froude model law is applied in the following fluid flow problems :

1. Free surface flows such as flow over spillways, weirs, sluices, channels etc.,
2. Flow of jet from an orifice or nozzle,
3. Where waves are likely to be formed on surface,
4. Where fluids of different densities flow over one another.

Let V_m = Velocity of fluid in model,

L_m = Linear dimension or length of model,

g_m = Acceleration due to gravity at a place where model is tested.

and V_P, L_P and g_P are the corresponding values of the velocity, length and acceleration due to gravity for the prototype. Then according to Froude model law,

$$(F_e)_{model} = (F_e)_{prototype} \text{ or } \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_P}{\sqrt{g_P L_P}} \qquad ...(12.18)$$

12.9.3 Euler's Model Law. Euler's model law is the law in which the models are designed on Euler's number which means for dynamic similarity between the model and prototype, the Euler number for model and prototype should be equal. Euler's model law is applicable when the pressure forces are alone predominant in addition to the inertia force. According to this law :

$$(E_u)_{model} = (E_u)_{prototype} \qquad ...(12.28)$$

If V_m = Velocity of fluid in model,

p_m = Pressure of fluid in model,

ρ_m = Density of fluid in model,

and V_P, p_P, ρ_P = Corresponding values in prototype, then

Substituting these values in equation (12.28), we get

$$\frac{V_m}{\sqrt{p_m/\rho_m}} = \frac{V_P}{\sqrt{p_P/\rho_P}} \qquad ...(12.29)$$

If fluid is same in model and prototype, then equation (12.29) becomes as

$$\frac{V_m}{\sqrt{p_m}} = \frac{V_P}{\sqrt{p_P}} \qquad ...(12.30)$$

12.9.4 Weber Model Law. Weber model law is the law in which models are based on Weber's number, which is the ratio of the square root of inertia force to surface tension force. Hence where surface tension effects predominate in addition to inertia force, the dynamic similarity between the model and prototype is obtained by equating the Weber number of the model and its prototype. Hence according to this law :

$$(W_e)_{model} = (W_e)_{prototype}, \qquad \text{where } W_e \text{ is Weber number and } = \frac{V}{\sqrt{\sigma / \rho L}}$$

If V_m = Velocity of fluid in model,

σ_m = Surface tensile force in model,

ρ_m = Density of fluid in model,

L_m = Length of surface in model,

and $V_P, \sigma_P, \rho_P, L_P$ = Corresponding values of fluid in prototype.

Then according to Weber law, we have

$$\frac{V_m}{\sqrt{\sigma_m / \rho_m L_m}} = \frac{V}{\sqrt{\sigma_P / \rho_P L_P}} \qquad \text{...(12.31)}$$

Weber model law is applied in following cases :

1. Capillary rise in narrow passages,
2. Capillary movement of water in soil,
3. Capillary waves in channels,
4. Flow over weirs for small heads.

12.9.5 Mach Model Law. Mach model law is the law in which models are designed on Mach number, which is the ratio of the square root of inertia force to elastic force of a fluid. Hence where the forces due to elastic compression predominate in addition to inertia force, the dynamic similarity between the model and its prototype is obtained by equating the Mach number of the model and its prototype. Hence according to this law :

$$(M)_{\text{model}} = (M)_{\text{prototype}}$$

where M = Mach number = $\dfrac{V}{\sqrt{K/\rho}}$

If V_m = Velocity of fluid in model,

K_m = Elastic stress for model,

ρ_m = Density of fluid in model,

and V_P, K_P and ρ_P = Corresponding values for prototype. Then according to Mach law,

$$= \frac{V_m}{\sqrt{K_m/\rho_m}} = \frac{V}{\sqrt{K_P/\rho_P}} \qquad \text{...(12.32)}$$